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Rapid Note

Truncated Lévy laws and 2D turbulence

B. Dubrulle[1,2,a] and J.-Ph. Laval[3]

[1] CNRS[b], CEA/DAPNIA/SAp L'Orme des Merisiers, 709, 91191 Gif-sur-Yvette, France
[2] CNRS[c], Observatoire Midi-Pyrénées, 14 avenue E. Belin, 31400 Toulouse, France
[3] CEA/DAPNIA/SAp L'Orme des Merisiers, 709, 91191 Gif-sur-Yvette, France

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Abstract. We study the probability distribution functions and scaling properties of truncated Lévy processes with sharp cut-offs. We find that they display features analog to those observed in some 2D numerical simulations of turbulence.

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1 Introduction

Lévy laws were introduced by Paul Lévy [2] in his search of stable laws (stable by convolution). They also arise in the context of generalization of the Central Limit Theorem, for variables with infinite variance. Lévy laws are characterized by one or two algebraic wings. They have therefore attracted special attention in systems where the probability distribution function displays an algebraic behavior in part of its wings, like financial markets [3] or more recently 2D turbulence [1,4]. The main physical explanation of the importance of Lévy laws in turbulence was given some times ago by Takayasu [5]. The idea is to decompose the velocity into contribution coming from individual vortices, with singular cores (see *e.g.* [6]). If the vortices are statistically independent and "fractal", the probability distribution of their properly renormalized sum is in the basin of attraction of a stable law, *i.e.* a Lévy law, whose index depending on the index of singularity of individual vortices.

There are actually no rigorous proof of the various hypotheses entering the explanation (existence of singular core, independence of the vortices, fractality of the turbulence), so the applicability of Lévy laws to turbulence should still be taken as speculative. Our goal is not here to try to prove or disprove the main ingredients of this theory, but to refine it slightly to make it compatible with the existence of physical cut-offs, which prevents velocity or vorticity to reach unlimited values. For example, Min *et al.* [1] use simulations with smoothened vortex cores, thereby regularizing the vorticity distribution and producing a natural cut-off in vorticity. The importance of these cut-offs is mentioned in the paper of Min *et al.* [1], and taken as responsible for the deviations from the algebraic behavior observed in the far tails of the distribution. We provide here further mathematical and numerical support of this explanation, in the light of the work of Mantegna and Stanley [7]. They observed that realistic systems cannot be represented by exact Lévy laws because of physical cut-offs. These cut-offs limit the sizes of the possible jumps of the variables, and then usually induce a finite variance, in contrast with ideal Lévy laws. This remark led them to introduce the concept of Truncated Lévy Flights, *i.e.* processes with properties resembling those of Lévy flights in a bounded area of parameter space, and with a finite variance. As such, they do not obey the stability criterion anymore and instead satisfy the Central Limit theorem: a sum of TLF converges towards a Gaussian process, albeit sometimes very slowly. In their paper, Mantegna and Stanley mainly discussed the probability of first return to zero, and the rapidity of convergence of the process towards a Gaussian distribution. They recently studied numerically some properties of the probability distribution function [8]. Their analysis was completed later by Koponen [9] who proposed another variant of Truncated Lévy flights, with a smoother exponential cut-off, which enables analytic computations and detailed analysis of the PDF. This new variant is however not fully satisfactory from a physical point of view because it does not present sharp cut-offs, and cannot represent processes in a box, for example. Motivated by this remark, we have decided to study a class of processes, with sharp cut-offs, but with continuous

[a] e-mail: bdubru@ariane.saclay.cea.fr
ou dubrulle@srvdec.obs-mip.fr
[b] URA 2052
[c] URA 285

distributions. This last property is a slight improvement with respect to the model of Mantegna and Stanley which is discontinuous at the boundaries. It does not change qualitatively the results obtained, but is more satisfactory form a physical point of view. We found that the sharp cut-offs indeed produce a sharp transition from algebraic decay to exponential decay in the wings of the distribution which bears many resemblance with the transition obtained in 2D turbulence. Our analysis clearly differs from that of Min *et al.* [1] since we introduce *a priori* distributions, instead of using probability distributions obtained through simulations of a set of vortices. Our goal indeed is not to "fit" the results obtained by Min *et al.*, but rather to see whether their conclusions are compatible with those obtained by the study of truncated Lévy laws.

2 The model

We consider random variables $\{z_i\}$ with identical distribution $f(z)$. We follow Koponen and define the truncated Lévy process as the limit of the sum of the random variables $\{z_i\}$ in the Poissonian stochastic process. It is characterized by a distribution $P(x)$, whose characteristic function obeys:

$$\ln \hat{P}(k) = -t \int_{-\infty}^{\infty} \left(1 - e^{-ikz}\right) f(z) dz. \tag{1}$$

For a smooth, exponential cut-off, $f(z)$ can be taken as [9]

$$f_{kopo}(z) = \begin{cases} A_- e^{-\lambda|z|}|z|^{-1-\nu}, & z < 0, \\ A_+ e^{-\lambda z} z^{-1-\nu}, & z \geq 0, \end{cases} \tag{2}$$

where λ^{-1} sets the cut-off value, $0 < \nu < 2$ and $A_\pm$ are constants characterizing the skewness of the distribution.

For a sharp but continuous cut-off, we take $f(z)$ as:

$$f_{DL}(z) = \begin{cases} 0, & z < \lambda_-^{-1}, \\ A|\text{arctanh}(\lambda_- z)|^{-1-\nu}, & \lambda_-^{-1} < z < 0, \\ A|\text{arctanh}(\lambda_+ z)|^{-1-\nu}, & 0 < z < \lambda_+^{-1}, \\ 0, & z > \lambda_+^{-1}. \end{cases} \tag{3}$$

The distribution $P(x)$ can then be computed by Fourier transform. It displays some identical features in both cases, stemming from the properties of the characteristic function. At small t, the integral defining $\ln \hat{P}(k)$ is dominated by the contribution due to $|z| \sim 0$, *i.e.* behaves like $-k^\nu$ for large k, exactly like for a Lévy distribution of parameter ν. The distribution $P(x)$ then displays algebraic tails, except far in the wings. At large t, the behavior of $\ln \hat{P}(k)$ is dominated by the behavior at large z of $f(z)$, which decreases faster than z^{-3}. The distribution then converges towards a Gaussian process.

3 Numerical results

3.1 PDF

We have used the expression of the characteristic function (1) to compute the probability distribution function $P(x,t)$ for various values of t by inverse Fourier transform. When the function $f_{DL}(z)$ (Eq. (3)) is used as input, one gets the results given in Figures 1 and 4. At small t, one observes a highly non Gaussian distribution, with a peak at the center separated from wide wings by a sharp cut-off. At large t, the distribution is smoother, with no peak at the center, and no large wings. These features confirm the analysis provided in Section 2 from the properties of the characteristic function: at small t, the central peak is actually the contribution from the "Lévy" part of the process. This is illustrated in Figure 2 where the central part of the PDF, with $\nu = 1$ is compared with the Lévy distribution of same index, *i.e.* the Cauchy distribution:

$$P(x) = \frac{1}{\pi} \frac{c}{c^2 + x^2} \cdot \tag{4}$$

The wings after the cut-off are exponential (inset of Fig. 2), exactly like in the numerical results of Min *et al.* [1]. At large t, the distribution is Gaussian, as shown in the comparison displayed in Figure 3. We have checked similar agreement with other values of ν, where the shape of the corresponding Lévy distribution cannot be given analytically. It is interesting to observe that the sharp cut-off obtained at small t is not present anymore when the PDF is computed from the prescription of Koponen (2) (inset of Fig. 3). On the other hand, we have checked that the sharp cut-off is still present when we used the (discontinuous) prescription proposed by Mantegna and Stanley, where $f(z)$ is equal to $z^{-1-\nu}$ over a bounded domain, and zero elsewhere. This shows that the cut-off observed in the

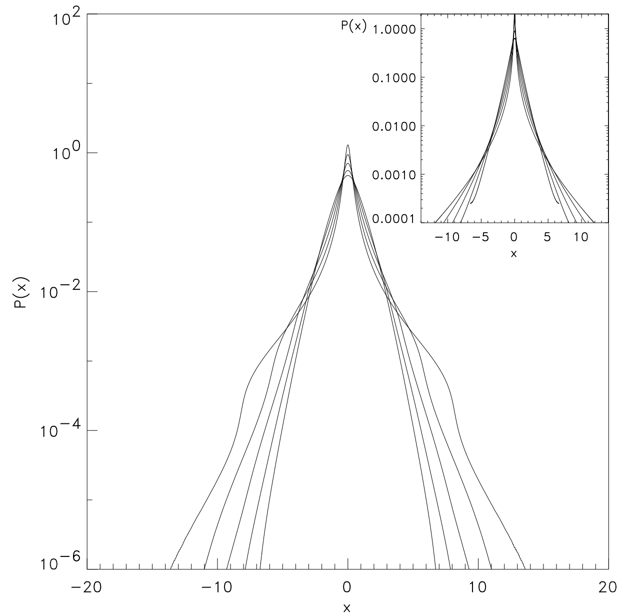

Fig. 1. Normalized distribution function $P(x)$ computed with $f_{DL}(z)$, $\nu = 1.0$, $\lambda_- = \lambda_+ = 0.05$, $A = 1$ and $t = \{0.2, 0.4, 0.8, 1.6, 3.2\}$ By comparison the normalized distribution function $P(x)$ computed with the formula of Koponen $f_{kopo}(z)$, $\nu = 0.5$, $\lambda = 0.05$, $A^+ = A^- = 1$ and $t = \{0.2, 0.4, 0.8, 1.6, 3.2\}$ is shown in the inset. There are no sharp cut-offs in this case.

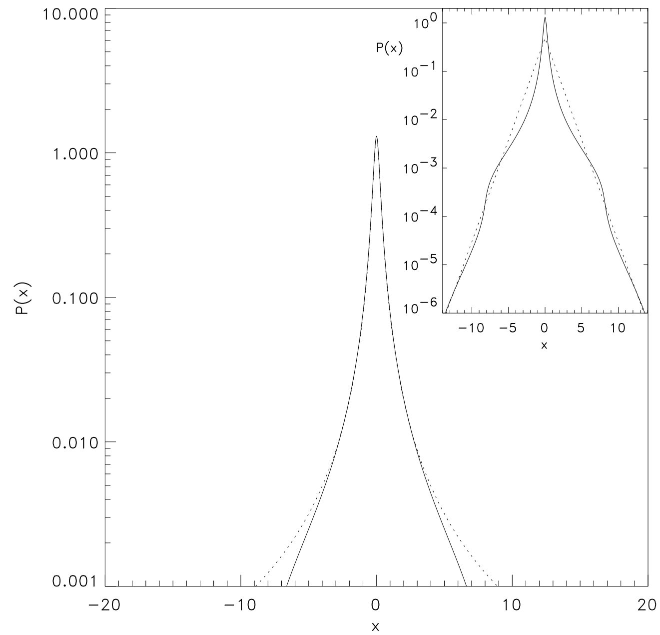

Fig. 2. Fit of the normalized distribution function $P(x)$ computed with $f_{DL}(z)$, $t = 0.2$, $\nu = 1$, $\lambda_- = \lambda_+ = 0.05$ (continuous line) by a Cauchy distribution (dotted line) define by (4) with $c = 0.25$. The inset shows the fit of the tails with an exponential distribution (dotted line).

PDF is the consequence on the sharp cut-off imposed on $f(z)$, and thus, reflects finite size effects. We note also that cut-offs have no reason to be symmetrical around $z = 0$. When they are symmetrical, the distribution is symmetrical, like in Figure 1; when they are not symmetrical, a slight dissymmetry appears on the PDF (Fig. 4).

3.2 Scaling exponents

Another interesting information about the variations with t of the PDF can be get by computing its moments, and their scaling properties. The scaling properties of the moments can be studied, like in turbulence, *via* the scaling exponents ζ_p defined as:

$$\zeta_p = \frac{d \ln \langle |x_t|^p \rangle}{d \ln t}\,. \tag{5}$$

An ideal Lévy distribution of index ν follows a self-similar type of behavior [3]:

$$P(x,t) = \frac{1}{t^{1/\nu}} P\left(\frac{x}{t^{1/\nu}},\ 1\right). \tag{6}$$

Using this property, it is easy to show that any moments follows:

$$\langle x^p \rangle = \alpha_p t^{p/\nu} \tag{7}$$

with

$$\alpha_p = \int_{-\infty}^{+\infty} y^p P(y,1) dy. \tag{8}$$

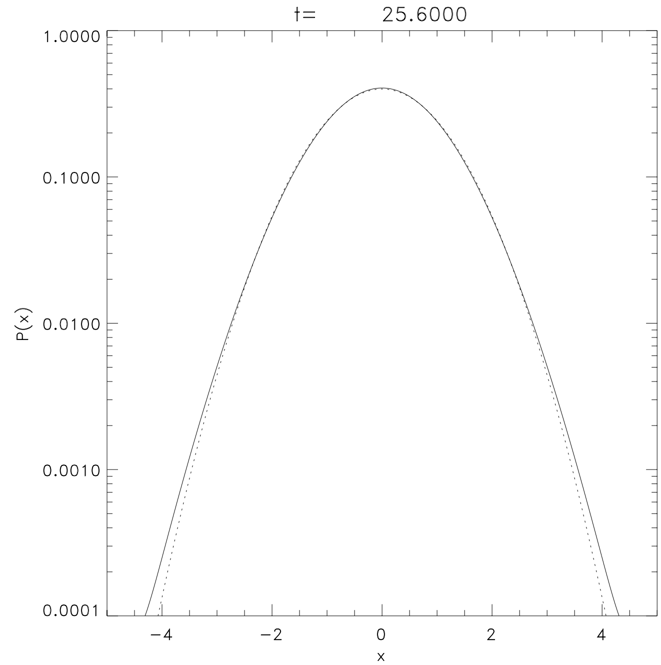

Fig. 3. Fit of the normalized distribution function $P(x)$ computed with $f_{DL}(z)$, $t = 25.6$, $\nu = 1$, $\lambda_- = \lambda_+ = 0.05$ (continuous line) by a Gaussian distribution (dotted line).

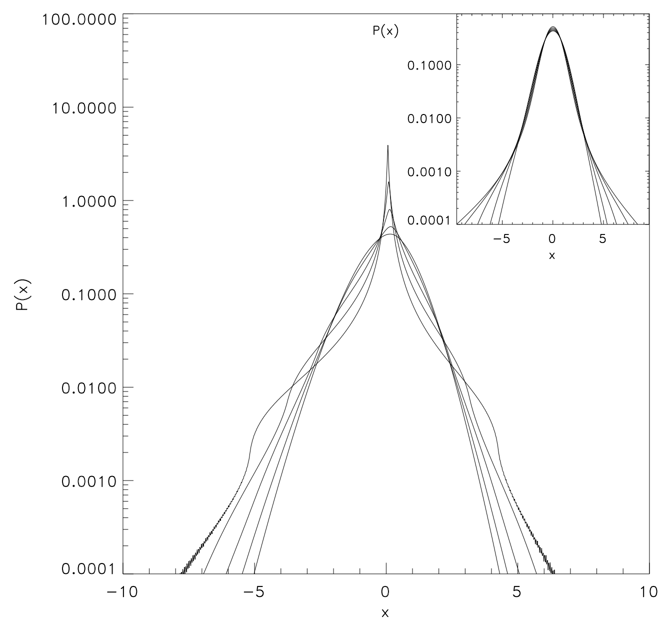

Fig. 4. Normalized distribution function $P(x)$ computed with $f_{DL}(z)$, $\nu = 0.5$, $\lambda_- = 0.04$, $\lambda_+ = 0.05$ and $t = \{0.2, 0.4, 0.8, 1.6, 3.2\}$ The inset shows the normalized distribution function $P(x)$ computed with $f_{DL}(z)$, for another set of parameters $\nu = 1.9$, $\lambda_- = 0.04$, $\lambda_+ = 0.05$ and $t = \{0.2, 0.4, 0.8, 1.6, 3.2\}$.

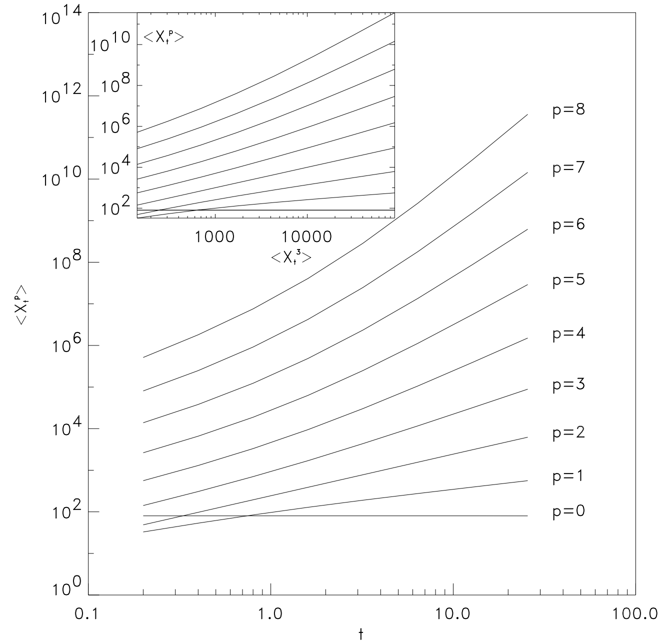

Fig. 5. Moments of order p ($\langle|x_t|^p\rangle$) as a function of t computed with $f_{DL}(z)$, $\nu = 1.$, $\lambda_- = \lambda_+ = 0.05$, $A = 1$ and $t = \{0.2, 0.4, 0.8, 1.6, 3.2\}$. The inset shows the same moments $\langle|x_t|^p\rangle$ as a function of the third moment $\langle|x_t|^3\rangle$.

For a Gaussian process, $\alpha_{2p+1} = 0$ and α_{2p} is finite. For a Lévy distribution with $0 < \nu < 2$, α_p, for $p > 2$, is infinite. However, in all cases, the local derivative (5) is finite, constant and equal to $\zeta_p = p/\nu$. In our case, all moments are convergent but the scaling exponents are not constant everywhere. However, for any value of ν we considered, we found that there is always a range of value of t where these local exponents are constant: this is the analog of an "inertial range" if t, which is related to the number of vortices, varies like a power law of the turbulent scale ℓ. The index of the power law if fixed by the condition that in the inertial range, the third order structure function varies like ℓ. This gives:

$$t \propto \ell^{1/\zeta_3}, \tag{9}$$

where ζ_3 is the scaling exponent computed using (5). Moreover, we also found that there is the analog of an Extended Self Similarity property: if we plot $\langle x^p\rangle$ as a function of, say, $\langle x^3\rangle$ (inset Fig. 5), we observe better defined scaling range than in the $\langle x^p\rangle$ *versus* t representation (Fig. 5). This means that relative scaling exponents ζ_p/ζ_3 are better defined. Also, they are the exact analog of the relative scaling exponents computed in turbulence, for the velocity differences structure functions. The results obtained for the PDF of Figures 4 and 1 are summarized in Table 1. For comparison, the values measured in the inverse cascade of a 2D numerical simulation [10] are also displayed. It can be seen that these relative exponents are similar to those obtained in 2D turbulence, and clearly differs from the value $\zeta_p/\zeta_3 = p/3$ which would be obtained if the distribution were a pure Gaussian or Lévy distribution. This can be seen as another manifestation of finite size effects, which induce a modification of the self-similar properties of ordinary Lévy laws. We can also note a slow convergence of the scaling exponents towards $p/3$ as the value of ν tends towards the Gaussian value.

Table 1. Comparison between relative scaling exponents in our PDFs and in 2D numerical simulations of [10].

ν	ζ_1/ζ_3	ζ_2/ζ_3	ζ_4/ζ_3	ζ_5/ζ_3	ζ_6/ζ_3	ζ_7/ζ_3	ζ_8/ζ_3
0.5	0.360	0.692	1.286	1.552	1.799	2.026	2.236
1.0	0.364	0.694	1.290	1.566	1.830	2.083	2.327
1.9	0.344	0.678	1.308	1.602	1.884	2.154	2.415
2D		0.7	1.27		1.73		

4 Summary

Inspired by the work of Mantegna and Stanley [7], we have studied the probability distribution functions of truncated Lévy processes, and their scaling properties. Our study shows that finite size effects affect the probability distribution function by inducing a sharp transition from algebraic to exponential decay at small times. We also studied the scaling properties of our truncated Lévy processes, by computing local and relative scaling exponents of the moments. We found the existence of an analog of an inertial range where local scaling exponents are constant, and an analog of the property of Extended Self Similarity. The relative scaling exponents display correction to the self-similarity which decrease as the probability distribution function gets closer to a Gaussian. These features are analog to those obtained in 2D turbulence by [4,1]. We thus propose that if 2D turbulence is to be described by something related to Lévy process, finite size effects should be taken into account, and truncated Lévy processes should be considered instead.

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